

Mock Tridiagonal Systems

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Abstract

We introduce the notion of a *mock tridiagonal system*. This is a generalization of a tridiagonal system in which the irreducibility assumption is replaced by a certain non-vanishing condition. We show how mock tridiagonal systems can be used to construct tridiagonal systems that meet certain specifications. This paper is part of our ongoing project to classify the tridiagonal systems up to isomorphism.

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1 Tridiagonal systems and mock tridiagonal systems

The concept of a tridiagonal system was introduced in [15, Definition 2.1] as a natural generalization of a Leonard system [36, 38, 39] and as a tool for studying $(P$ and $Q)$ -polynomial association schemes [2, 7, 34, 35]. One can view the concept as part of the bispectral problem [12, 13, 41]. There are connections to representation theory [1, 11, 14, 17–22, 24, 37] and statistical mechanical models [3–6, 8–10, 25]. More results on tridiagonal systems can be found in [15, 16, 23, 26–33]. It remains an open problem to classify the tridiagonal systems up to isomorphism, but classifications do exist for some special cases [18, 21, 22, 24, 33, 40]. To make further progress on the classification problem, in this paper we introduce the notion of a *mock tridiagonal system*. This is a generalization of a tridiagonal system in which the irreducibility assumption is replaced by a certain nonvanishing condition. In our main result, we show how mock tridiagonal systems can be used to construct tridiagonal systems that meet certain specifications.

Before going into more detail we recall the definition of a tridiagonal system. We will use the following terms. Throughout this paper \mathbb{F} denotes a field, and V denotes a vector space over \mathbb{F} with finite positive dimension. Let $\text{End}(V)$ denote the \mathbb{F} -algebra of all \mathbb{F} -linear transformations from V to V . Given $A \in \text{End}(V)$ and a subspace $W \subseteq V$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case θ is the *eigenvalue* of A associated with W . We say that A is *diagonalizable*

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whenever V is spanned by the eigenspaces of A . Assume A is diagonalizable. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of A . For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ ($0 \leq j \leq d$). Here I denotes the identity of $\text{End}(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $I = \sum_{i=0}^d E_i$; (ii) $E_i E_j = \delta_{i,j} E_i$ ($0 \leq i, j \leq d$); (iii) $V_i = E_i V$ ($0 \leq i \leq d$); (iv) $A = \sum_{i=0}^d \theta_i E_i$. Moreover

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}. \quad (1)$$

Note that each of $\{A^i\}_{i=0}^d$, $\{E_i\}_{i=0}^d$ is a basis for the \mathbb{F} -subalgebra of $\text{End}(V)$ generated by A . Moreover $\prod_{i=0}^d (A - \theta_i I) = 0$.

Definition 1.1 [15, Definition 2.1] By a *tridiagonal system* (or *TD system*) on V we mean a sequence $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^\delta)$ that satisfies (i)–(iv) below.

(i) Each of A, A^* is a diagonalizable element of $\text{End}(V)$.

(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A such that

$$E_i A^* E_j = 0 \quad \text{if } |i - j| > 1, \quad (0 \leq i, j \leq d).$$

(iii) $\{E_i^*\}_{i=0}^\delta$ is an ordering of the primitive idempotents of A^* such that

$$E_i^* A E_j^* = 0 \quad \text{if } |i - j| > 1, \quad (0 \leq i, j \leq \delta).$$

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say that Φ is *over* \mathbb{F} .

Note 1.2 According to a common notational convention X^* denotes the conjugate-transpose of X . We are not using this convention. For the TD system in Definition 1.1 the linear transformations A, E_i, A^*, E_i^* are arbitrary subject to (i)–(iv) above.

Definition 1.3 Referring to the TD system Φ in Definition 1.1, it turns out that the integers d and δ are equal [15, Lemma 4.5]; we call this common value the *diameter* of Φ .

In the theory of TD systems the following situation often occurs: we wish to show that there exists a TD system that meets some given specifications [24, 33]. Suppose we have a candidate $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$, and we just want to check that it really is a TD system. It is usually routine to verify conditions (i)–(iii) of Definition 1.1, but often difficult to verify condition (iv) of Definition 1.1. In this paper we give a method for constructing TD systems that overcomes this difficulty. The method is based on the notion of a mock tridiagonal system which we now introduce.

Definition 1.4 By a *mock tridiagonal system* (or *MTD system*) on V we mean a sequence $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ that satisfies (i)–(iv) below.

(i) Each of A, A^* is a diagonalizable element of $\text{End}(V)$.

(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A such that

$$E_i A^* E_j = 0 \quad \text{if } |i - j| > 1, \quad (0 \leq i, j \leq d).$$

(iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^* such that

$$E_i^* A E_j^* = 0 \quad \text{if } |i - j| > 1, \quad (0 \leq i, j \leq d).$$

(iv) Each of $E_0^* E_0 E_0^*, E_0^* E_d E_0^*$ is nonzero.

We say that Φ is *over* \mathbb{F} .

Lemma 1.5 *Any TD system on V is an MTD system on V .*

Proof: Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote the TD system in question. We show that each of $E_0^* E_0 E_0^*, E_0^* E_d E_0^*$ is nonzero. By [30, Lemma 5.1] the map $E_0^* V \rightarrow E_0 V, u \mapsto E_0 u$ is bijective and the map $E_0 V \rightarrow E_0^* V, v \mapsto E_0^* v$ is bijective. By construction E_0^* acts as the identity on $E_0^* V$. Therefore the restriction of $E_0^* E_0 E_0^*$ to $E_0^* V$ gives a bijection $E_0^* V \rightarrow E_0^* V$. The space $E_0^* V$ is nonzero so $E_0^* E_0 E_0^*$ is nonzero. One similarly finds that $E_0^* E_d E_0^*$ is nonzero. \square

In Section 4 we display some MTD systems that are not TD systems.

2 Statement of the main theorem

In this section we state our main results. In order to do this concisely we first discuss some basic concepts.

Definition 2.1 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote an MTD system on V . For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with the eigenspace $E_i V$ (resp. $E_i^* V$). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ . Observe that $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in \mathbb{F} .

Referring to Definition 2.1, we call Φ *sharp* whenever $E_0^* V$ has dimension 1 [30, Definition 1.5]. By [32, Theorem 1.3], every TD system over an algebraically closed field is sharp.

The following notation will be useful.

Definition 2.2 Let λ denote an indeterminate and let $\mathbb{F}[\lambda]$ denote the \mathbb{F} -algebra consisting of the polynomials in λ that have all coefficients in \mathbb{F} . Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ denote scalars in \mathbb{F} . For $0 \leq i \leq d$ we define the following polynomials in $\mathbb{F}[\lambda]$:

$$\begin{aligned}\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \\ \eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}), \\ \tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*), \\ \eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*).\end{aligned}$$

Note that each of τ_i , η_i , τ_i^* , η_i^* is monic with degree i .

Lemma 2.3 Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a sharp MTD system on V , with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Then for $0 \leq i \leq d$ there exists a unique $\zeta_i \in \mathbb{F}$ such that

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}.$$

Proof: The given MTD system is sharp so E_0^*V has dimension 1. Therefore E_0^* has rank 1 and this implies $E_0^* \mathcal{A} E_0^* = \mathbb{F} E_0^*$, where $\mathcal{A} = \text{End}(V)$. The result follows. \square

Definition 2.4 Let Φ denote a sharp MTD system. By the *split sequence* of Φ we mean the sequence $\{\zeta_i\}_{i=0}^d$ from Lemma 2.3. Observe that $\zeta_0 = 1$.

Definition 2.5 Let Φ denote a sharp MTD system. By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ and $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ .

The following proposition indicates the importance of the parameter array. The proposition refers to the notion of isomorphism for TD systems, which is defined in [30, Section 3].

Proposition 2.6 [32, Theorem 1.6] *Two sharp TD systems over \mathbb{F} are isomorphic if and only if they have the same parameter array.*

We now state our main result.

Theorem 2.7 *Let Φ denote a sharp MTD system over \mathbb{F} . Then there exists a sharp TD system over \mathbb{F} that has the same parameter array as Φ .*

We will use the following strategy to prove Theorem 2.7. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a sharp MTD system on V , with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$. Let T denote the \mathbb{F} -subalgebra of $\text{End}(V)$ generated by A, A^* , and consider the T -module TE_0^*V . We show that TE_0^*V contains a unique maximal proper T -submodule. Denote this submodule by M and consider the quotient T -module $L = TE_0^*V/M$. By construction the T -module L is nonzero and irreducible. We show that the sequence $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ acts on L as a sharp TD system with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$.

3 The proof of the main theorem

In this section we give a proof of Theorem 2.7. Throughout this section we fix a sharp MTD system $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ on V .

Definition 3.1 Let T denote the \mathbb{F} -subalgebra of $\text{End}(V)$ generated by A, A^* . By definition T contains the identity I of $\text{End}(V)$. By (1) the algebra T contains each of E_i, E_i^* for $0 \leq i \leq d$.

We now consider the T -module TE_0^*V . We will be discussing proper T -submodules of TE_0^*V . The word *proper* means that the T -submodule in question is properly contained in TE_0^*V , or in other words not equal to TE_0^*V .

Definition 3.2 Let W denote a proper T -submodule of TE_0^*V . Then W is called *maximal* whenever W is not contained in any proper T -submodule of TE_0^*V , besides itself.

Our first goal is to show that TE_0^*V has a unique maximal proper T -submodule.

Lemma 3.3 *Let W denote a proper T -submodule of TE_0^*V . Then $E_0^*W = 0$.*

Proof: Suppose $E_0^*W \neq 0$. The space E_0^*V contains E_0^*W and has dimension 1, so $E_0^*V = E_0^*W$. The space W is T -invariant and $E_0^* \in T$ so $E_0^*W \subseteq W$. Therefore $E_0^*V \subseteq W$, which yields $TE_0^*V \subseteq W$. This contradicts the fact that W is properly contained in TE_0^*V . Therefore $E_0^*W = 0$. \square

Lemma 3.4 *Let W and W' denote proper T -submodules of TE_0^*V . Then $W + W'$ is a proper T -submodule of TE_0^*V .*

Proof: We show $W + W' \neq TE_0^*V$. Define $K = \{u \in TE_0^*V \mid E_0^*u = 0\}$. Then $K \neq TE_0^*V$, since $0 \neq E_0^*V \subseteq TE_0^*V$ and E_0^* acts as the identity on E_0^*V . The space K contains each of W, W' by Lemma 3.3, so K contains $W + W'$. Therefore $W + W' \neq TE_0^*V$ and the result follows. \square

Lemma 3.5 *There exists a unique maximal proper T -submodule of TE_0^*V .*

Proof: Concerning existence, let W denote the subspace of TE_0^*V spanned by all the proper T -submodules of TE_0^*V . Then W is a proper T -submodule of TE_0^*V by Lemma 3.4, and since TE_0^*V has finite dimension. The T -submodule W is maximal by the construction. Concerning uniqueness, suppose W and W' are maximal proper T -submodules of TE_0^*V . By Lemma 3.4 $W + W'$ is a proper T -submodule of TE_0^*V . The space $W + W'$ contains each of W, W' so $W + W'$ is equal to each of W, W' by the maximality of W and W' . Therefore $W = W'$ and the result follows. \square

Definition 3.6 Let M denote the maximal proper T -submodule of TE_0^*V . Let L denote the quotient T -module TE_0^*V/M . By construction the T -module L is nonzero and irreducible.

Proposition 3.7 *The sequence $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ acts on L as a TD system with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$.*

Proof: The following equations hold in T and hence on L :

$$E_i E_j = \delta_{i,j} E_i, \quad E_i^* E_j^* = \delta_{i,j} E_i^* \quad (0 \leq i, j \leq d), \quad (2)$$

$$I = \sum_{i=0}^d E_i, \quad I = \sum_{i=0}^d E_i^*, \quad (3)$$

$$A = \sum_{i=0}^d \theta_i E_i, \quad A^* = \sum_{i=0}^d \theta_i^* E_i^*, \quad (4)$$

$$E_i A^* E_j = 0 \quad \text{if } |i - j| > 1 \quad (0 \leq i, j \leq d), \quad (5)$$

$$E_i^* A E_j^* = 0 \quad \text{if } |i - j| > 1 \quad (0 \leq i, j \leq d). \quad (6)$$

Define $S = \{i | 0 \leq i \leq d, E_i L \neq 0\}$ and $S^* = \{i | 0 \leq i \leq d, E_i^* L \neq 0\}$. By the equations on the left in (2), (3) we have $L = \sum_{i \in S} E_i L$ (direct sum). Using the equations on the left in (2), (4) we find that for $i \in S$ the space $E_i L$ is an eigenspace for A with eigenvalue θ_i . By these comments A is diagonalizable on L with eigenvalues $\{\theta_i\}_{i \in S}$. By the equation on the left in (2), for $i \in S$ the element E_i acts as the identity on $E_i L$ and vanishes on $E_j L$ for $j \neq i$ ($j \in S$). In other words E_i acts on L as the primitive idempotent of A associated with θ_i . Similarly A^* is diagonalizable on L with eigenvalues $\{\theta_i^*\}_{i \in S^*}$, and for $i \in S^*$ the element E_i^* acts on L as the primitive idempotent of A^* associated with θ_i^* . We now show that there exist nonnegative integers r, k ($r + k \leq d$) such that $S = \{r, r + 1, \dots, r + k\}$. The set S is nonempty since L is nonzero and equal to $\sum_{i \in S} E_i L$. Define $r = \min\{i | i \in S\}$ and $p = \max\{i | i \in S\}$. For $r + 1 \leq i \leq p - 1$ we have $i \in S$; otherwise $E_r L + \dots + E_{i-1} L$ is a nonzero proper T -submodule of L , contradicting the irreducibility of the T -module L . Now $S = \{r, r + 1, \dots, r + k\}$ where $k = p - r$. Similarly there exist nonnegative integers t, k^* ($t + k^* \leq d$) such that $S^* = \{t, t + 1, \dots, t + k^*\}$. By the argument so far, the sequence $(A; \{E_i\}_{i=r}^{r+k}; A^*; \{E_i^*\}_{i=t}^{t+k^*})$ acts on L as a TD system. For this system we invoke the first sentence of Definition 1.3 to get $k = k^*$. We now show that $r = 0$. Suppose $r \neq 0$. Then $0 \notin S$ so $E_0 L = 0$. This implies $E_0 TE_0^* V \subseteq M$ so $E_0 E_0^* V \subseteq M$. In this containment we apply E_0^* to both sides and use $E_0^* M = 0$ to get $E_0^* E_0 E_0^* V = 0$. This contradicts Definition 1.4(iv) so $r = 0$. Next we show that $t = 0$. Suppose $t \neq 0$. Then $0 \notin S^*$ so $E_0^* L = 0$. This implies $E_0^* TE_0^* V \subseteq M$ so $E_0^* V \subseteq M$. But then $TE_0^* V \subseteq M$ since M is T -invariant. This contradicts the fact that M is properly contained in $TE_0^* V$, so $t = 0$. We now show that $k = d$. Suppose $k \neq d$. Then $d \notin S$ so $E_d L = 0$. This implies $E_d TE_0^* V \subseteq M$ so $E_d E_0^* V \subseteq M$. In this containment we apply E_0^* to both sides and use $E_0^* M = 0$ to get $E_0^* E_d E_0^* V = 0$. This contradicts Definition 1.4(iv) so $k = d$. We have shown $(r, t, k) = (0, 0, d)$, so now $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ acts on L as a TD system which we denote by Φ . By construction Φ has eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By Lemma 2.3 and Definition 2.4 the equations

$$E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)} \quad (0 \leq i \leq d)$$

hold on V . Therefore these equations hold in T and hence on L . By this and Definition 2.4 the sequence $\{\zeta_i\}_{i=0}^d$ is the split sequence for Φ . By these comments Φ has parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ and the result follows. \square

Theorem 2.7 is immediate from Proposition 3.7.

We finish this section with two corollaries of Theorem 2.7 and Proposition 3.7. The first corollary is about the dimensions of the \mathbb{F} -vector spaces L and M from Definition 3.6.

Corollary 3.8 *The following (i), (ii) hold.*

- (i) $\dim(L) \geq d + 1$;
- (ii) $\dim(M) \leq \dim(TE_0^*V) - d - 1$.

Proof: (i) By Proposition 3.7 there exists a TD system on L that has diameter d .
(ii) Recall that L is the quotient TE_0^*V/M so $\dim(L) + \dim(M) = \dim(TE_0^*V)$. The result follows from this and (i) above. \square

In the next corollary we list some constraints satisfied by the parameter array of a sharp MTD system.

Corollary 3.9 *Let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ denote the parameter array of a sharp MTD system. Then (i)–(iii) hold below:*

- (i) $\theta_i \neq \theta_j, \theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leq i, j \leq d$).
- (ii) $\zeta_0 = 1, \zeta_d \neq 0$, and

$$0 \neq \sum_{i=0}^d \eta_{d-i}(\theta_0) \eta_{d-i}^*(\theta_0^*) \zeta_i.$$

- (iii) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d - 1$.

Proof: Let Φ denote the sharp MTD system in question. Then Φ satisfies condition (i) by the last sentence of Definition 2.1. Φ satisfies condition (ii) by Theorem 2.7 and since (ii) holds for any sharp TD system with parameter array $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ [30, Corollary 8.3]. Φ satisfies condition (iii) by Theorem 2.7 and since (iii) holds for any TD system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$ [15, Theorem 11.1]. \square

4 An example

In this section we consider a family of sharp MTD systems over \mathbb{F} that have diameter 2. In [40] Vidar described the members of this family that are TD systems. Our focus here is on the family members that are not TD systems. For these members we find the space TE_0^*V from below Definition 3.1, and the space M from Definition 3.6. We also describe the induced TD system on $L = TE_0^*V/M$, from Proposition 3.7. Throughout this section we make use of the work of Vidar [40, Section 9].

From now on we fix a sequence

$$(\{\theta_i\}_{i=0}^2; \{\theta_i^*\}_{i=0}^2; \{\zeta_i\}_{i=0}^2) \quad (7)$$

of scalars in \mathbb{F} that satisfy (i), (ii) below.

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ ($0 \leq i, j \leq 2$).

(ii) $\zeta_0 = 1$, $\zeta_2 \neq 0$, and

$$0 \neq \zeta_2 + \zeta_1(\theta_0 - \theta_2)(\theta_0^* - \theta_2^*) + (\theta_0 - \theta_1)(\theta_0 - \theta_2)(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*). \quad (8)$$

Our first goal is to display an MTD system over \mathbb{F} that has parameter array (7).

Definition 4.1 Let V denote the vector space \mathbb{F}^4 (column vectors). Define

$$A = \begin{pmatrix} \theta_0 & 0 & 0 & 0 \\ 1 & \theta_1 & 0 & 0 \\ 0 & 0 & \theta_1 & 0 \\ 0 & 1 & \zeta_1^\times & \theta_2 \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & \zeta_1 & \zeta_2 & 0 \\ 0 & \theta_1^* & 0 & 0 \\ 0 & 0 & \theta_1^* & 1 \\ 0 & 0 & 0 & \theta_2^* \end{pmatrix},$$

where

$$\zeta_1^\times = \zeta_1 + (\theta_0 - \theta_1)(\theta_0^* - \theta_1^*) - (\theta_1 - \theta_2)(\theta_1^* - \theta_2^*). \quad (9)$$

We view $A, A^* \in \text{End}(V)$.

Lemma 4.2 *The matrix A (resp. A^*) is diagonalizable with eigenvalues $\{\theta_i\}_{i=0}^2$ (resp. $\{\theta_i^*\}_{i=0}^2$). For $0 \leq i \leq 2$ the dimension of the eigenspace for A (resp. A^*) associated with θ_i (resp. θ_i^*) is $\binom{2}{i}$.*

Proof: One checks that A has characteristic polynomial $(\lambda - \theta_0)(\lambda - \theta_1)^2(\lambda - \theta_2)$ and minimal polynomial $(\lambda - \theta_0)(\lambda - \theta_1)(\lambda - \theta_2)$. Our assertions for A follow from this. Our assertions for A^* are similarly proved. \square

Definition 4.3 For $0 \leq i \leq 2$ let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) associated with θ_i (resp. θ_i^*).

Lemma 4.4 *We have*

$$\begin{aligned}
E_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{\theta_0 - \theta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{(\theta_0 - \theta_1)(\theta_0 - \theta_2)} & 0 & 0 & 0 \end{pmatrix}, & E_0^* &= \begin{pmatrix} 1 & \frac{\zeta_1}{\theta_0^* - \theta_1^*} & \frac{\zeta_2}{\theta_0^* - \theta_1^*} & \frac{\zeta_2}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{\theta_1 - \theta_0} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{(\theta_1 - \theta_0)(\theta_1 - \theta_2)} & \frac{1}{\theta_1 - \theta_2} & \frac{\zeta_1^\times}{\theta_1 - \theta_2} & 0 \end{pmatrix}, & E_1^* &= \begin{pmatrix} 0 & \frac{\zeta_1}{\theta_1^* - \theta_0^*} & \frac{\zeta_2}{\theta_1^* - \theta_0^*} & \frac{\zeta_2}{(\theta_1^* - \theta_0^*)(\theta_1^* - \theta_2^*)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{\theta_1^* - \theta_2^*} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{(\theta_2 - \theta_0)(\theta_2 - \theta_1)} & \frac{1}{\theta_2 - \theta_1} & \frac{\zeta_1^\times}{\theta_2 - \theta_1} & 1 \end{pmatrix}, & E_2^* &= \begin{pmatrix} 0 & 0 & 0 & \frac{\zeta_2}{(\theta_2^* - \theta_0^*)(\theta_2^* - \theta_1^*)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\theta_2^* - \theta_1^*} \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Proof: The matrices $\{E_i\}_{i=0}^2$ are obtained using (1). One similarly obtains $\{E_i^*\}_{i=0}^2$. \square

Lemma 4.5 *The following (i)–(iii) hold.*

- (i) $E_0 A^* E_2 = 0, E_2 A^* E_0 = 0;$
- (ii) $E_0^* A E_2^* = 0, E_2^* A E_0^* = 0;$
- (iii) $E_0^* E_0 E_0^* \neq 0, E_0^* E_2 E_0^* \neq 0.$

Proof: (i), (ii) Routine calculation using the matrices in Definition 4.1 and Lemma 4.4.
(iii) The $(1, 1)$ -entry of $E_0^* E_0 E_0^*$ is $(\theta_0 - \theta_1)^{-1}(\theta_0 - \theta_2)^{-1}(\theta_0^* - \theta_1^*)^{-1}(\theta_0^* - \theta_2^*)^{-1}$ times the expression on the right in (8). This expression is nonzero so $E_0^* E_0 E_0^* \neq 0$. The $(1, 1)$ -entry of $E_0^* E_2 E_0^*$ is $(\theta_2 - \theta_0)^{-1}(\theta_2 - \theta_1)^{-1}(\theta_0^* - \theta_1^*)^{-1}(\theta_0^* - \theta_2^*)^{-1}$ times ζ_2 . By assumption $\zeta_2 \neq 0$ so $E_0^* E_2 E_0^* \neq 0$. \square

Proposition 4.6 *The sequence $(A; \{E_i\}_{i=0}^2; A^*; \{E_i^*\}_{i=0}^2)$ is an MTD system on V with parameter array $(\{\theta_i\}_{i=0}^2; \{\theta_i^*\}_{i=0}^2; \{\zeta_i\}_{i=0}^2)$.*

Proof: Let Φ denote the sequence in question. To show that Φ is an MTD system on V , we verify the conditions (i)–(iv) of Definition 1.4. Condition (i) holds by Lemma 4.2. Condition (ii) holds by Definition 4.3 and Lemma 4.5(i). Condition (iii) holds by Definition 4.3 and Lemma 4.5(ii). Condition (iv) holds by Lemma 4.5(iii). We have verified the conditions of Definition 1.4, so Φ is an MTD system on V . By Lemma 4.2 and Definition 4.3, Φ has eigenvalue sequence $\{\theta_i\}_{i=0}^2$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^2$. Using Lemma 2.3 and Definition 2.4 we find that Φ has split sequence $\{\zeta_i\}_{i=0}^2$. By these comments Φ has parameter array $(\{\theta_i\}_{i=0}^2; \{\theta_i^*\}_{i=0}^2; \{\zeta_i\}_{i=0}^2)$ and the result follows. \square

Proposition 4.7 (Vidar [40, Theorem 9.1]) *The following (i), (ii) are equivalent:*

- (i) *The MTD system from Proposition 4.6 is a TD system;*
- (ii) $\zeta_1 \zeta_1^\times \neq \zeta_2$.

In Proposition 4.6 we displayed an MTD system $(A; \{E_i\}_{i=0}^2; A^*; \{E_i^*\}_{i=0}^2)$. In what follows we consider the corresponding algebra T from Definition 3.1, and the T -modules L, M from Definition 3.6.

Lemma 4.8 *Assume $\zeta_1 \zeta_1^\times = \zeta_2$. Then the following (i), (ii) hold.*

- (i) $TE_0^*V = V$.
- (ii) *The \mathbb{F} -vector space M is one-dimensional and spanned by $(0, -\zeta_1^\times, 1, 0)^t$.*

Proof: (i) The span of the vector $(1, 0, 0, 0)^t$ is E_0^*V , the span of $(0, 0, 0, 1)^t$ is E_2V , the span of $(0, 1, 0, 0)^t$ is $(A - \theta_0 I)E_0^*V$, and the span of $(0, 0, 1, 0)^t$ is $(A^* - \theta_2^* I)E_2V$. Therefore

$$V = E_0^*V + (A - \theta_0 I)E_0^*V + (A^* - \theta_2^* I)E_2V + E_2V \quad (\text{direct sum}). \quad (10)$$

By the form of A ,

$$E_2V = (A - \theta_1 I)(A - \theta_0 I)E_0^*V. \quad (11)$$

Combining (10), (11) we find $TE_0^*V = V$.

(ii) Let W denote the subspace of V spanned by the vector $(0, -\zeta_1^\times, 1, 0)^t$. Using Definition 4.1 one checks that $(A - \theta_1 I)W = 0$ and $(A^* - \theta_1^* I)W = 0$. Therefore W is a T -submodule of V . Of course W is properly contained in V so $W \subseteq M$. Consequently the dimension of M is at least one. The dimension of M is at most one by Corollary 3.8(ii), so the dimension of M is one. The result follows. \square

Definition 4.9 Assume $\zeta_1 \zeta_1^\times = \zeta_2$. By Lemma 4.8(ii) the \mathbb{F} -vector space L has a basis $\{v_i\}_{i=0}^2$ such that

$$\begin{aligned} v_0 &= (1, 0, 0, 0)^t + M, \\ v_1 &= (0, 1, 0, 0)^t + M, \\ v_2 &= (0, 0, 0, 1)^t + M. \end{aligned}$$

Proposition 4.10 *Assume $\zeta_1 \zeta_1^\times = \zeta_2$. With respect to the basis $\{v_i\}_{i=0}^2$ from Definition 4.9 the matrices representing A, A^* are as follows.*

$$A : \begin{pmatrix} \theta_0 & 0 & 0 \\ 1 & \theta_1 & 0 \\ 0 & 1 & \theta_2 \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \zeta_1 & 0 \\ 0 & \theta_1^* & \zeta_1^\times \\ 0 & 0 & \theta_2^* \end{pmatrix}.$$

Proof: Routine using Definition 4.1 and Definition 4.9. \square

Proposition 4.11 Assume $\zeta_1 \zeta_1^\times = \zeta_2$. With respect to the basis $\{v_i\}_{i=0}^2$ from Definition 4.9 the matrices representing $\{E_i\}_{i=0}^2$ and $\{E_i^*\}_{i=0}^2$ are as follows.

$$\begin{aligned} E_0 : & \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\theta_0 - \theta_1} & 0 & 0 \\ \frac{1}{(\theta_0 - \theta_1)(\theta_0 - \theta_2)} & 0 & 0 \end{pmatrix}, & E_0^* : & \begin{pmatrix} 1 & \frac{\zeta_1}{\theta_0^* - \theta_1^*} & \frac{\zeta_2}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ E_1 : & \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\theta_1 - \theta_0} & 1 & 0 \\ \frac{1}{(\theta_1 - \theta_0)(\theta_1 - \theta_2)} & \frac{1}{\theta_1 - \theta_2} & 0 \end{pmatrix}, & E_1^* : & \begin{pmatrix} 0 & \frac{\zeta_1}{\theta_1^* - \theta_0^*} & \frac{\zeta_2}{(\theta_1^* - \theta_0^*)(\theta_1^* - \theta_2^*)} \\ 0 & 1 & \frac{\zeta_1^\times}{\theta_1^* - \theta_2^*} \\ 0 & 0 & 0 \end{pmatrix}, \\ E_2 : & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{(\theta_2 - \theta_0)(\theta_2 - \theta_1)} & \frac{1}{\theta_2 - \theta_1} & 1 \end{pmatrix}, & E_2^* : & \begin{pmatrix} 0 & 0 & \frac{\zeta_2}{(\theta_2^* - \theta_0^*)(\theta_2^* - \theta_1^*)} \\ 0 & 0 & \frac{\zeta_1^\times}{\theta_2^* - \theta_1^*} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Proof: Routine using Lemma 4.4 and Definition 4.9. □

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